

# MULTIPLICATIVE INVERSES IN SHORT INTERVALS

S. BAIER

## 1. INTRODUCTION AND RESULTS

T.D. Browning and A. Haynes [1] considered the following problem. Let  $p$  be a prime,  $J$  be an integer and  $I_1^{(j)}, I_2^{(j)}$  with  $1 \leq j \leq J$  be finite sequences of subintervals of  $(0, p)$ . Under which conditions is there a  $j$  such that

$$xy \equiv 1 \pmod{p}, \quad (x, y) \in \left(I_1^{(j)} \times I_2^{(j)}\right) \cap \mathbb{Z}^2 \quad (1)$$

has a solution? They proved the following.

**Theorem 1.** *Let  $H, K > 0$  and let  $I_1^{(j)}, I_2^{(j)} \subseteq (0, p)$  be subintervals, for  $1 \leq j \leq J$ , such that*

$$\left|I_1^{(j)}\right| = H \quad \text{and} \quad \left|I_2^{(j)}\right| = K$$

and

$$I_1^{(j)} \cap I_1^{(k)} = \emptyset \quad \text{for all } j \neq k. \quad (2)$$

Then there exists  $j \in \{1, \dots, J\}$  for which (1) has a solution if

$$J \gg \frac{p^3 \log^4 p}{H^2 K^2}. \quad (3)$$

The proof of this theorem in [1] relies on the following new mean value theorem for short Kloosterman sums by Browning and Haynes.

**Theorem 2.** *If  $I_1, \dots, I_J \subseteq (0, p)$  are disjoint subintervals, with  $H/2 \leq |I_j| \leq H$  for each  $j$ , then for any  $l \in (\mathbb{Z}/p\mathbb{Z})^*$ , we have*

$$\sum_{j=1}^J \left| \sum_{n \in I_j} e\left(\frac{l\bar{n}}{p}\right) \right|^2 \leq 2^{12} p \log^2 H.$$

In this note, we prove Theorem 1, in a slightly generalized and refined form, by a different method which doesn't use Theorem 2 but Poisson summation and Weil's estimate for Kloosterman sums. Moreover, we improve this result under certain additional conditions on the spacing of the intervals  $I_i^{(j)}$ .

We note that we could add the assumption

$$H \geq \log p \quad (4)$$

---

2000 *Mathematics Subject Classification.* 11N25.

to Theorem 1 without weakening the result because  $J \ll p/H$  under the conditions of this theorem which contradicts (3) if  $H < \log p$  since  $K \leq p$ . We want to assume (4) throughout the sequel. Moreover, we want to assume without loss of generality that the intervals  $I_1^{(j)}$  and  $I_2^{(j)}$  are closed and centered at integers  $N_j$  and  $M_j$ , respectively. We prove the following.

**Theorem 3.** *Assume that all conditions in Theorem 1 are satisfied, except possibly (2). Assume the integers  $M_j$  are  $X$ -spaced modulo  $p$ , i.e.*

$$p \cdot \left\| \frac{M_j - M_k}{p} \right\| \geq X \quad \text{for all } j, k \text{ with } j \neq k,$$

for some  $X \geq 1$ , where  $\|z\|$  denotes the distance of the real number  $z$  to the nearest integer. Then there exists  $j \in \{1, \dots, J\}$  for which (1) has a solution if

$$J \gg \frac{p^3 \log^{3+\varepsilon} p}{HK^2 \min\{H, X\}}. \quad (5)$$

Clearly, Theorem 3 implies Theorem 1. If the intervals  $I_1^{(j)}$  and  $I_2^{(j)}$  are equispaced, respectively, we obtain an improvement of the result above, provided that the intervals  $I_1^{(j)}$  are not spaced too far away.

**Theorem 4.** *Assume that all conditions in Theorem 1 are satisfied. Assume further that the integers  $M_j$  and  $N_j$  lie in arithmetic progression, respectively, i.e.*

$$M_j = M + jX \quad \text{and} \quad N_j = N + jY$$

for certain integers  $M, N, X, Y$  and all  $j \in \{1, \dots, J\}$ . Then there exists  $j \in \{1, \dots, J\}$  for which (1) has a solution if

$$X \ll \frac{HK}{p^{1/2}(\log p)^{1+\varepsilon}} \quad \text{and} \quad J \gg \frac{p^{3/2} \log^{2+\varepsilon} p}{HK}. \quad (6)$$

## 2. BASIC APPROACH

Set

$$w(t) := \exp(-\pi t^2).$$

This has Fourier transform

$$\hat{w}(t) = w(t) = \exp(-\pi t^2). \quad (7)$$

Set

$$x := \frac{H}{(\log p)^{1/2+\varepsilon}}, \quad y := \frac{K}{(\log p)^{1/2+\varepsilon}} \quad (8)$$

and

$$T := \sum_{j=1}^J \sum_{|m-M_j| \leq H/2} \sum_{\substack{|n-N_j| \leq K/2 \\ m \equiv n \pmod{p}}} w\left(\frac{m-M_j}{x}\right) w\left(\frac{n-N_j}{y}\right),$$

where here and in the sequel, we assume that  $(n, p) = 1$ , and  $\bar{n}$  denotes the multiplicative inverse of  $n$  modulo  $p$ . Then clearly, (1) has a solution if  $T > 0$ . Now the general strategy is to extend the sums over  $m$  and  $n$  to all integers and use Poisson summation and Weil's estimate for Kloosterman sums.

We write

$$T = S - S_1 - S_2, \quad (9)$$

where

$$S := \sum_{j=1}^J \sum_{\substack{m \\ m \equiv \bar{n} \pmod{p}}} \sum_n w\left(\frac{m - M_j}{x}\right) w\left(\frac{n - N_j}{y}\right),$$

$$S_1 := \sum_{j=1}^J \sum_{\substack{|m - M_j| > H/2 \\ m \equiv \bar{n} \pmod{p}}} \sum_n w\left(\frac{m - M_j}{x}\right) w\left(\frac{n - N_j}{y}\right)$$

and

$$S_2 := \sum_{j=1}^J \sum_{\substack{|m - M_j| \leq H/2 \\ |n - N_j| > K/2 \\ m \equiv \bar{n} \pmod{p}}} w\left(\frac{m - M_j}{x}\right) w\left(\frac{n - N_j}{y}\right).$$

From  $J, H, K \leq p$  ( $J \leq p$  following from  $X \geq 1$ ), (7) and (8), it is evident that  $S_1$  and  $S_2$  are negligible, i.e.

$$S_1, S_2 \ll_A p^{-A} \quad \text{for any } A > 0. \quad (10)$$

In the following section, we estimate the sum  $S$ .

### 3. APPLICATION OF POISSON SUMMATION

We have

$$\begin{aligned} S &= \sum_{j=1}^J \sum_{\substack{m \\ m \equiv \bar{n} \pmod{p}}} \sum_n w\left(\frac{m - M_j}{x}\right) w\left(\frac{n - N_j}{y}\right) \\ &= \frac{1}{p} \sum_{k=-(p-1)/2}^{(p-1)/2} \sum_{j=1}^J \sum_m \sum_n w\left(\frac{m - M_j}{x}\right) w\left(\frac{n - N_j}{y}\right) e\left(k \cdot \frac{m - \bar{n}}{p}\right) \\ &= \frac{1}{p} J \sum_m w\left(\frac{m}{x}\right) \sum_n w\left(\frac{n}{y}\right) + \\ &\quad \frac{1}{p} \sum_{\substack{k=-(p-1)/2 \\ k \neq 0}}^{(p-1)/2} \sum_{j=1}^J \sum_m w\left(\frac{m - M_j}{x}\right) e\left(k \cdot \frac{m}{p}\right) \sum_n w\left(\frac{n - N_j}{y}\right) e\left(-k \cdot \frac{\bar{n}}{p}\right). \end{aligned}$$

Using Poisson summation, the terms in the last line can be transformed as follow. First,

$$\frac{1}{p} J \sum_m w\left(\frac{m}{x}\right) \sum_n w\left(\frac{n}{y}\right) = \frac{Jxy}{p} \cdot \hat{w}(0)^2.$$

Second,

$$\begin{aligned} & \sum_m w\left(\frac{m - M_j}{x}\right) e\left(k \cdot \frac{m}{p}\right) = \sum_m w\left(\frac{m}{x}\right) e\left(k \cdot \frac{m + M_j}{p}\right) \\ & = x \cdot e\left(k \cdot \frac{M_j}{p}\right) \cdot \sum_{m \equiv k \pmod p} \hat{w}\left(\frac{mx}{p}\right) = x \cdot e\left(k \cdot \frac{M_j}{p}\right) \cdot F_k(x), \quad \text{say.} \end{aligned}$$

Third,

$$\begin{aligned} & \sum_n w\left(\frac{n - N_j}{y}\right) e\left(-k \cdot \frac{n}{p}\right) = \sum_{c \pmod p} e\left(-k \cdot \frac{\bar{c}}{p}\right) \sum_{n \equiv c + N_j \pmod p} w\left(\frac{n}{y}\right) \\ & = \frac{y}{p} \cdot \sum_{c \pmod p} e\left(-k \cdot \frac{\bar{c}}{p}\right) \sum_l \hat{w}\left(\frac{ly}{p}\right) e\left(l \cdot \frac{c + N_j}{p}\right) \\ & = \frac{y}{p} \cdot \sum_l \hat{w}\left(\frac{ly}{p}\right) e\left(l \cdot \frac{N_j}{p}\right) S(l, -k; p), \end{aligned}$$

where

$$S(l, -k; p) = \sum_{c=1}^{p-1} e\left(\frac{lc - k\bar{c}}{p}\right)$$

is the Kloosterman sum. Putting the above together, we get

$$\begin{aligned} S &= \frac{Jxy}{p} \cdot \hat{w}(0)^2 + \\ & \frac{xy}{p^2} \cdot \sum_{\substack{k=-(p-1)/2 \\ k \neq 0}}^{(p-1)/2} \sum_l S(l, -k; p) F_k(x) \hat{w}\left(\frac{ly}{p}\right) \sum_{j=1}^J e\left(\frac{kM_j + lN_j}{p}\right). \end{aligned} \quad (11)$$

We note that if  $-(p-1)/2 \leq k \leq (p-1)/2$ , then

$$\begin{aligned} F_k(x) &= \sum_{m \equiv k \pmod p} \hat{w}\left(\frac{mx}{p}\right) = \sum_{r \in \mathbb{Z}} \hat{w}\left(\left(\frac{k}{p} + r\right)x\right) \\ &\ll \exp\left(-\frac{|k|}{p} \cdot x\right) \cdot \sum_{r=0}^{\infty} \exp(-rx) \ll \exp\left(-\frac{|k|}{p} \cdot x\right) \end{aligned} \quad (12)$$

since  $x \geq 1$  if  $\varepsilon \leq 1/2$  by (4) and (8).

## 4. PROOF OF THEOREM 3

By Weil's bound for Kloosterman sums, we have

$$S(l, -k; p) \ll p^{1/2} \quad \text{if } k \not\equiv 0 \pmod{p}. \quad (13)$$

Using the Cauchy-Schwarz inequality and (12), it follows that

$$\begin{aligned} & \sum_{\substack{k=-(p-1)/2 \\ k \neq 0}}^{(p-1)/2} \sum_l S(l, -k; p) F_k(x) \hat{w} \left( \frac{ly}{p} \right) \sum_{j=1}^J e \left( \frac{kM_j + lN_j}{p} \right) \\ & \ll p^{1/2} \left( \sum_{\substack{k=-(p-1)/2 \\ k \neq 0}}^{(p-1)/2} \sum_l F_k(x) \hat{w} \left( \frac{ly}{p} \right) \right)^{1/2} \\ & \quad \left( \sum_{k=-(p-1)/2}^{(p-1)/2} \sum_l F_k(x) \hat{w} \left( \frac{ly}{p} \right) \left| \sum_{j=1}^J e \left( \frac{kM_j + lN_j}{p} \right) \right|^2 \right)^{1/2} \\ & \ll \frac{p^{3/2}}{(xy)^{1/2}} \left( \sum_{k=-(p-1)/2}^{(p-1)/2} \sum_l F_k(x) \hat{w} \left( \frac{ly}{p} \right) \left| \sum_{j=1}^J e \left( \frac{kM_j + lN_j}{p} \right) \right|^2 \right)^{1/2}. \end{aligned} \quad (14)$$

Expanding the square, we get

$$\begin{aligned} & \sum_{k=-(p-1)/2}^{(p-1)/2} \sum_l F_k(x) \hat{w} \left( \frac{ly}{p} \right) \left| \sum_{j=1}^J e \left( \frac{kM_j + lN_j}{p} \right) \right|^2 \\ & = \sum_{j_1, j_2=1}^J \left( \sum_{k=-(p-1)/2}^{(p-1)/2} F_k(x) e \left( k \cdot \frac{M_{j_1} - M_{j_2}}{p} \right) \right) \left( \sum_l \hat{w} \left( \frac{ly}{p} \right) e \left( l \cdot \frac{N_{j_1} - N_{j_2}}{p} \right) \right). \end{aligned}$$

Using (12), we have

$$\sum_{k=-(p-1)/2}^{(p-1)/2} F_k(x) e \left( k \cdot \frac{M_{j_1} - M_{j_2}}{p} \right) \ll \sum_{k=-(p-1)/2}^{(p-1)/2} F_k(x) \ll \frac{p}{x}.$$

Similarly,

$$\sum_l \hat{w} \left( \frac{ly}{p} \right) e \left( l \cdot \frac{N_{j_1} - N_{j_2}}{p} \right) \ll \sum_l \hat{w} \left( \frac{ly}{p} \right) \ll \frac{p}{y}.$$

Moreover, removing the weight function  $F_k(x)$  using partial summation and using the familiar estimate for geometric sums, we have

$$\sum_{k=-(p-1)/2}^{(p-1)/2} F_k(x) e \left( k \cdot \frac{M_{j_1} - M_{j_2}}{p} \right) \ll \left\| \frac{M_{j_1} - M_{j_2}}{p} \right\|^{-1}.$$

It follows that

$$\begin{aligned} & \sum_{k=-(p-1)/2}^{(p-1)/2} \sum_l F_k(x) \hat{w} \left( \frac{ly}{p} \right) \left| \sum_{j=1}^J e \left( \frac{kM_j + lN_j}{p} \right) \right|^2 \\ & \ll \frac{p}{y} \sum_{j_1, j_2=1}^J \min \left\{ \left\| \frac{M_{j_1} - M_{j_2}}{p} \right\|^{-1}, \frac{p}{x} \right\}. \end{aligned} \quad (15)$$

Using the fact that the  $M_j$ 's are  $X$ -spaced modulo  $p$ , we obtain

$$\begin{aligned} & \ll \sum_{j_1, j_2=1}^J \min \left\{ \left\| \frac{M_{j_1} - M_{j_2}}{p} \right\|^{-1}, \frac{p}{x} \right\} \\ & \ll J \sum_{j=0}^{J-1} \min \left\{ \frac{p}{jX}, \frac{p}{x} \right\} \\ & \ll Jp \left( \frac{1}{x} + \frac{\log 2J}{X} \right). \end{aligned} \quad (16)$$

Combining (9), (10), (11), (14), (15) and (16), we arrive at

$$T = \frac{Jxy}{p} \cdot \hat{w}(0)^2 + O \left( (\log 2J)^{1/2} (Jxp)^{1/2} \left( \frac{1}{x} + \frac{1}{X} \right)^{1/2} \right). \quad (17)$$

For the right-hand side to be greater 0 (i.e. error term < main term), it suffices that

$$J \gg \frac{p^3 \log p}{xy^2} \left( \frac{1}{x} + \frac{1}{X} \right),$$

which holds if (5) is satisfied. This implies Theorem 3.

## 5. PROOF OF THEOREM 4

Under the conditions of Theorem 4, we have

$$\begin{aligned} \sum_{j=1}^J e \left( \frac{kM_j + lN_j}{p} \right) &= e \left( \frac{kM + lN}{p} \right) \sum_{j=1}^J e \left( \frac{j(kX + lY)}{p} \right) \\ &\ll \min \left\{ \left\| \frac{kX + lY}{p} \right\|^{-1}, J \right\}. \end{aligned} \quad (18)$$

Using (13) and (18), we obtain

$$\begin{aligned} & \sum_{k=-(p-1)/2}^{(p-1)/2} \sum_l S(l, -k; p) F_k(x) \hat{w}\left(\frac{ly}{p}\right) \sum_{j=1}^J e\left(\frac{kM_j + lN_j}{p}\right) \\ & \ll p^{1/2} \sum_l \hat{w}\left(\frac{ly}{p}\right) \sum_{k=-(p-1)/2}^{(p-1)/2} F_k(x) \min\left\{\left\|\frac{kX + lY}{p}\right\|^{-1}, J\right\}. \end{aligned} \quad (19)$$

We estimate the inner-most sum over  $k$  by

$$\begin{aligned} \sum_{k=-(p-1)/2}^{(p-1)/2} F_k(x) \min\left\{\left\|\frac{kX + lY}{p}\right\|^{-1}, J\right\} & \ll \frac{p/x}{p/X} \cdot \left(J + \frac{p}{X} \cdot \log p\right) \\ & = \frac{X}{x} \cdot J + \frac{p}{x} \cdot \log p, \end{aligned}$$

where we use (12) and  $X \geq x$  (which follows from (2)). Hence, we get

$$\begin{aligned} & p^{1/2} \sum_l \hat{w}\left(\frac{ly}{p}\right) \sum_{k=-(p-1)/2}^{(p-1)/2} F_k(x) \min\left\{\left\|\frac{kX + lY}{p}\right\|^{-1}, J\right\} \\ & \ll \frac{p^{3/2}}{y} \cdot \left(\frac{X}{x} \cdot J + \frac{p}{x} \cdot \log p\right). \end{aligned} \quad (20)$$

Combining (9), (10), (11), (19) and (20), we get

$$T = \frac{Jxy}{p} \cdot \hat{w}(0)^2 + O\left(\frac{X}{p^{1/2}} \cdot J + p^{1/2} \log p\right). \quad (21)$$

For the right-hand side to be greater 0, it suffices that

$$X \ll \frac{xy}{p^{1/2}} \quad \text{and} \quad J \gg \frac{p^{3/2} \log p}{xy},$$

which holds if (6) is satisfied. This implies Theorem 4.

## REFERENCES

- [1] T. Browning, A. Haynes, *Incomplete Kloosterman sums and multiplicative inverses in short intervals*, arXiv:1204.6374v1. 1, 1

S. BAIER, SCHOOL OF MATHEMATICS, UNIVERSITY OF EAST ANGLIA, NORWICH, NR4 7TJ, ENGLAND

*E-mail address:* s.baier@uea.ac.uk